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On the Generally Invariant Lagrangians for the Metric Field and Other Tensor Fields

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The Krupka and Trautman method for the description of all generally invariant functions of the components of geometrical object fields is applied to the invariants of second degree of the metrical field and other tensor fields. The complete system of differential identities fulfilled by the invariants mentioned is found and it is proved that these invariants depend on the tensor quantities only.

1. INTRODUCTION

Krupka and Trautman have given a general method for the description of all generally invariant functions of the components of geometrical objects and their derivatives of arbitrary degree (Krupka and Trautman, 1974; Krupka, 1974). Later on this method was successfully applied to the invariants of second degree of the metrical tensor field (Krupka, 1976; Krupka, 1978). Hereby the full system of differential identities fulfilled by the invariants mentioned has been written. The maximal number of functionally independent invariants has been found and some concrete basis of this system has been constructed. An analogical program has been also realized for invariants of the first degree of the components of metrical tensor and connection (Horák and Krupka, 1978). The physical meaning of invariants consists in the fact that they play the role of Lagrangians in the theories based on validity of variational principle, such as, for example, Einstein's theory in the first and the Einstein–Cartan theory in the second case.

The object of the present paper is to study further possibilities of the Krupka–Trautman method, especially for the situation arising in general relativity in the presence of nongravitational fields. We are concerned with invariants depending not only on metrical field, but also on some other tensor fields and their derivatives up to second degree. After an examination of the problem we ended by drawing the following conclusion.

The above-mentioned invariants depend on the tensor quantities only —i.e., on the metrical tensor, on the curvature tensor, and on some other tensor fields and their covariant derivatives of the first and of the second degree. Thus we managed to reduce the whole problem of the finding of invariants to the problem of finding invariants of given tensor which can be solved by the known classical methods (Dieudonné and Carrell, 1971). As a further corollary of the conclusion mentioned we are able to determine the number of functionally independent invariants.

Let us note that the present work stands in close connection with the quoted works of Krupka (Krupka, 1976; Krupka, 1978); and for that reason we do not need to occupy ourselves in detail with the theory there already explained.

2. FUNDAMENTAL STRUCTURES

We shall study invariant functions of fibered manifolds with the basis of dimension n and with typical fiber which is the Cartesian product of the member $T_n^2(R^{n*} \odot R^{n*})$ (corresponding to the metrical tensor field and its derivatives up to the second degree), arbitrary number of members $T_n^2(R)$ (corresponding to scalar fields), and arbitrary number of members $T_n^2(R^{n*} \otimes \cdots \otimes R^{n*})$ (corresponding to the covariant tensor fields of arbitrary degree). The presence of the metric makes it possible for us to take into consideration merely covariant fields.

For brevity let us limit ourselves to the case of fundamental structures for the determination of generally invariant Lagrangians dependent on the metrical field and on some vector field. (The general case of an arbitrary number of tensor fields will be discussed later.) In the given case a typical fiber will be

$$Q = T_n^2(R^{n*} \odot R^{n*}) \times T_n^2(R^{n*})$$

Let us denote canonical coordinates on $T_n^2(R^{n*} \odot R^{n*})$ as g_{ij}, $g_{ij,k}$, $g_{ij,kl}$ and canonical coordinates on $T_n^2(R^{n*})$ as A_i, $A_{i,j}$, $A_{i,jk}$. On the manifold Q the transformation group L_n^3 of all invertible 3-jets with the source and the target in the point $0 \in R^n$ acts in a natural way.

In conformity with Krupka's work (1978) let us introduce on

$T_n^2(R^{n*} \odot R^{n*})$ new coordinates $\tilde{g}_{ij}$, $\Gamma_{i,jk}$, R_{ijkl}, $S_{i,jkl}$, where

$$\tilde{g}_{ij} = g_{ij}, \qquad \Gamma_{i,jk} = \tfrac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i})$$
$$R_{ijkl} = \tfrac{1}{2}(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}) + \tfrac{1}{4}g^{mn}[(g_{mj,k} + g_{mk,j} - g_{jk,m})(g_{ni,l} + g_{nl,i} - g_{il,n}) - (g_{mj,l} + g_{ml,j} - g_{jl,m})(g_{ni,k} + g_{nk,i} - g_{ik,n})]$$
$$S_{i,jkl} = \tfrac{1}{3}(g_{ij,kl} + g_{il,jk} + g_{ik,jl}) - \tfrac{1}{6}(g_{jk,il} + g_{jl,ik} + g_{kl,ij})$$

To obtain a full system of coordinates on Q let us add new coordinates $\tilde{A}_i$, $A_{i,j}$, $A_{i;(j;k)}$, where

$$\tilde{A}_i = A_i, \qquad A_{i;j} = A_{i,j} - \tfrac{1}{2}g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l})A_k$$
$$A_{i;(j;k)} = A_{i,jk} - \tfrac{1}{2}\{g^{no}[\delta_k^m(g_{oi,j} + g_{oj,i} - g_{ij,o}) + \delta_j^m(g_{oi,k} + g_{ok,i} - g_{ik,o})] + g^{mo}\delta_i^n(g_{oj,k} + g_{ok,j} - g_{jk,o})\}A_{m,m}$$
$$+ \tfrac{1}{8}\{g^{lo}g^{mn}[(3g_{om,k} + g_{ok,m} - g_{km,o})(g_{ni,j} + g_{nj,i} - g_{ij,n}) + (3g_{om,j} + g_{oj,m} - g_{mj,o})(g_{ni,k} + g_{nk,i} - g_{ik,n}) + 2(g_{oi,m} + g_{om,i} - g_{im,o})(g_{nj,k} + g_{nk,j} - g_{jk,n})] - 2g^{ml}(2g_{mi,jk} + g_{mj,ik} + g_{mk,ij} - g_{ij,mk} - g_{ik,mj})\}A_l$$

Here g^{ij} is defined by the relations $g^{ij}g_{jk} = \delta_k^i$. Consequently $A_{i;j}$ are usual "covariant derivatives" of vector field and $A_{i;(j;k)}$ is symmetrized sum of the second "covariant derivatives."

Inverse transformations are (in the neighborhood of the points where $\det g_{ij} \neq 0$)

$$g_{ij} = \tilde{g}_{ij}, \qquad g_{ij,k} = \Gamma_{i,jk} + \Gamma_{j,ik}$$
$$g_{ij,kl} = S_{i,jkl} + S_{j,ikl} - \tfrac{1}{3}(R_{ikjl} + R_{jkil}) + \tfrac{1}{3}g^{mn}(\Gamma_{m,jk}\Gamma_{n,il} + \Gamma_{m,jl}\Gamma_{n,ik} - 2\Gamma_{m,kl}\Gamma_{n,ij})$$
$$A_i = \tilde{A}_i, \qquad A_{i,j} = A_{i;j} + g^{kl}\Gamma_{l,ij}A_k$$
$$A_{i,jk} = A_{i;(j;k)} + [g^{no}(\Gamma_{o,ik}\delta_j^m + \Gamma_{o,ij}\delta_k^m) + g^{mo}\Gamma_{o,jk}\delta_i^n]A_{m;n} + \tfrac{1}{6}g^{lm}[6S_{m,ijk} - R_{ijmk} - R_{ikmj} - 2g^{no}(\Gamma_{o,kj}\Gamma_{n,im} + \Gamma_{o,mj}\Gamma_{n,ik} + \Gamma_{o,km}\Gamma_{n,ij})]A_l$$

3. FUNDAMENTAL VECTOR FIELDS

In this part we shall study the natural action of L_n^3 on Q. We shall find the full system of differential identities for the invariants of this action.

Let us denote canonical coordinates on L_n^3 as a_j^i, a_{jk}^i, a_{jkl}^i. Let us write the action of the group L_n^3 on Q in the coordinates $\tilde{g}_{ij}$, $\Gamma_{i,jk}$, R_{ijkl}, $S_{i,jkl}$,

$\tilde{A}_i$, $A_{i;j}$, $A_{i;(j;k)}$. It holds that

$$\tilde{A}'_i = a_i{}^j\tilde{A}_j, \qquad A'_{i;j} = a_i{}^k a_j{}^l A_{k;l}, \qquad A'_{i;(j;k)} = a_i{}^l a_j{}^m a_k{}^n A_{l;(m;n)}$$
$$\tilde{g}'_{ij} = a_i{}^k a_j{}^l \tilde{g}_{kl}, \qquad \Gamma'_{i,jk} = a_i{}^l a_j{}^m a_k{}^n \Gamma_{l,mn} + a_i{}^l a^m_{jk}\tilde{g}_{lm}$$
$$R'_{ijkl} = a_i{}^m a_j{}^n a_k{}^o a_l{}^p R_{mnop}$$
$$\begin{aligned} S'_{i,jkl} = a_i{}^m a_j{}^n a_k{}^o a_l{}^p S_{m,nop} + \{a_i{}^m(a^o_{jl}a_k{}^n + a^o_{jk}a_l{}^n + a^o_{lk}a_j{}^n) \\ + \tfrac{1}{3}(a^m_{il}a_j{}^n a_k{}^o + a^m_{ik}a_j{}^n a_l{}^o + a^m_{ij}a_l{}^n a_k{}^o) \\ + \tfrac{1}{3}a_i{}^n(a^m_{jl}a_k{}^o + a^m_{jk}a_l{}^o + a^m_{lk}a_j{}^o)\}\Gamma_{m,no} \\ + [a^m_{jkl}a_i{}^n + \tfrac{1}{3}(a^m_{il}a^n_{jk} + a^m_{ik}a^n_{jl} + a^m_{ij}a^n_{kl})]\tilde{g}_{mn} \end{aligned}$$

The fundamental vector fields corresponding to presented relations are characterized in Krupka's work (1978). Let us denote them as $\Sigma_i{}^j$, Σ_i^{jk}, Σ_i^{jkl}. It holds for these fields

$$\Sigma_i{}^j = \Xi_i{}^j + \psi_i{}^j$$

where

$$\Xi_i{}^j = \frac{\partial g'_{kl}}{\partial a_j{}^i}\frac{\partial}{\partial g_{kl}} + \frac{\partial \Gamma'_{k,lm}}{\partial a_j{}^i}\frac{\partial}{\partial \Gamma_{k,lm}} + \frac{\partial R'_{klmn}}{\partial a_j{}^i}\frac{\partial}{\partial R_{klmn}} + \frac{\partial S'_{k,lmn}}{\partial a_j{}^i}\frac{\partial}{\partial S_{k,lmn}}$$

$$\Psi_i{}^j = \frac{\partial A'_k}{\partial a_j{}^i}\frac{\partial}{\partial A_k} + \frac{\partial A'_{k;l}}{\partial a_j{}^i}\frac{\partial}{\partial A_{k;l}} + \frac{\partial A'_{k;(l;m)}}{\partial a_j{}^i}\frac{\partial}{\partial A_{k;(l;m)}}$$

$$\Sigma_i^{jk} = \Xi_i^{jk} = \frac{\partial \Gamma'_{l,mn}}{\partial a^i_{jk}}\frac{\partial}{\partial \Gamma_{l,mn}} + \frac{\partial S'_{l,mno}}{\partial a^i_{jk}}\frac{\partial}{\partial S_{l,mno}}$$

$$\Sigma_i^{jkl} = \Xi_i^{jkl} = \frac{\partial S'_{m,nop}}{\partial a^i_{jkl}}\frac{\partial}{\partial S_{m,nop}}$$

(the derivatives are considered in the unit point $e = j_o{}^3 id \in L_n{}^3$; i.e., after their calculation we put $a_i{}^j = \delta_i{}^j$, $a^k_{ij} = 0$, $a^l_{ijk} = 0$.

The quantities $\Xi_i{}^j$, Ξ_i^{jk}, Ξ_i^{jkl} were calculated in Krupka (1978). For $\psi_i{}^j$ we obtain

$$\psi_i{}^j = A_i\frac{\partial}{\partial A_j} + A_{i;k}\frac{\partial}{\partial A_{j;k}} + A_{k;i}\frac{\partial}{\partial A_{k;j}} + A_{i;(k;l)}\frac{\partial}{\partial A_{j;(k;l)}} + 2A_{k;(i;l)}\frac{\partial}{\partial A_{k;(j;l)}} \tag{3.1}$$

It would be suitable to introduce new fields

$$\Sigma^{i,jkl} = g^{im}\Sigma_m^{jkl} = \Xi^{i,jkl}$$
$$\Sigma^{i,jk} = g^{il}\left(\Sigma_l^{jk} - \frac{\partial S'_{m,nop}}{\partial a^l_{jk}}\Sigma^{m,nop}\right) = \Xi^{i,jk}$$
$$\Sigma^+_{ij} = \tfrac{1}{2}(\tilde{\Sigma}_{i,j} + \tilde{\Sigma}_{j,i}), \qquad \Sigma^-_{ij} = \tfrac{1}{2}(\tilde{\Sigma}_{i,j} - \tilde{\Sigma}_{j,i})$$

where

$$\tilde{\Sigma}_{i,j} = -g_{ik}g_{jl}\Sigma^{k,l}$$

$$\Sigma^{k,l} = \tfrac{1}{2} g^{km}\left(\Sigma_m{}^l - \frac{\partial \Gamma'_{n,op}}{\partial a_l{}^m} \Sigma^{n,op} - \frac{\partial S'_{n,opq}}{\partial a_l{}^m} \Sigma^{n,opq}\right)$$

$$= \Xi^{k,l} + \tfrac{1}{2} g^{km}\psi_m{}^l$$

It follows therefore that

$$\Sigma_{ij}^{+} = \Xi_{ij}^{+} - \tfrac{1}{4}(g_{jk}\psi_i{}^k + g_{ik}\psi_j{}^k)$$

$$\Sigma_{ij}^{-} = \Xi_{ij}^{-} + \tfrac{1}{4}(g_{ik}\psi_j{}^k - g_{jk}\psi_i{}^k)$$

[$\Xi^{i,jkl}$, $\Xi^{i,jk}$, Ξ_{ij}^{+}, Ξ_{ij}^{-} are calculated in Krupka's work, $\psi_i{}^j$ is given by (3.1)].

The theorem following from our considerations runs therefore as follows:

In the neighborhood of every regular point of the manifold $Q = T_n{}^2(R^{n*} \odot R^{n*}) \times T_n{}^2(R^{n*})$ the Lie algebra $e_{L^3{}_n}(Q)$ is spanned on the vector fields

$$\Sigma^{i,jkl} = \frac{\partial}{\partial S_{i,jkl}}, \qquad \Sigma^{i,jk} = \frac{\partial}{\partial \Gamma_{i,jk}}$$

$$\Sigma_{ij}^{+} = \frac{\partial}{\partial g_{ij}} - (g_{in}R_{jklm} + g_{jn}R_{iklm})\frac{\partial}{\partial R_{nklm}} - \tfrac{1}{4}(g_{ik}\psi_j{}^k + g_{jk}\psi_i{}^k) \quad (3.2)$$

$$\Sigma_{ij}^{-} = (g_{in}R_{jklm} - g_{jn}R_{iklm})\frac{\partial}{\partial R_{nklm}} + \tfrac{1}{4}(g_{ik}\psi_j{}^k - g_{jk}\psi_i{}^k)$$

Every generally invariant function $\mathscr{L}$ defined on some open $L_n{}^3$ invariant neighborhood of a regular point of the manifold Q fulfils the system of equations

$$\Sigma^{i,jkl}(\mathscr{L}) = 0, \qquad \Sigma^{i,jk}(\mathscr{L}) = 0, \qquad \Sigma_{ij}^{+}(\mathscr{L}) = 0, \qquad \Sigma_{ij}^{-}(\mathscr{L}) = 0 \quad (3.3)$$

and—on the contrary—every function that fulfils this system is generally invariant.

It means that every generally invariant function $\mathscr{L}$ depends merely on g_{ij}, R_{ijkl}, A_i, $A_{i;j}$, $A_{i;(j;k)}$, i.e., on tensorial quantities only.

4. DISCUSSION

In the case of an arbitrary number of tensor fields of arbitrary degree it is again possible to introduce instead of the canonical coordinates $T_{a_1\cdots a_n}$, $T_{a_1\cdots a_n,i}$, $T_{a_1\cdots a_n,ij}$ new coordinates $\tilde{T}_{a_1\cdots a_n} = T_{a_1\cdots a_n}$, $T_{a_1\cdots a_n;i}$, $T_{a_1\cdots a_n;(i;j)}$ for every tensor field. In the transformation relations for these quantities the group $L_n{}^3$ acts by means of its components $a_j{}^i$ only. All the previous relations and conclusions remain valid with one exception, namely, the expression for

ψ_i^k will become in general more complicated. For example, for the tensor field of second degree it will be

$$\psi_i^j = A_{ik}\frac{\partial}{\partial A_{jk}} + A_{ki}\frac{\partial}{\partial A_{kj}} + A_{ik;l}\frac{\partial}{\partial A_{jk;l}} + A_{ki;l}\frac{\partial}{\partial A_{kj;l}} + A_{kl;i}\frac{\partial}{\partial A_{kl;j}}$$
$$+ A_{ik;(l;m)}\frac{\partial}{\partial A_{jk;(l;m)}} + A_{ki;(l;m)}\frac{\partial}{\partial A_{kj;(l;m)}} + 2A_{kl;(i;m)}\frac{\partial}{\partial A_{kl;(j;m)}} \quad (4.1)$$

and so on. Particularly, the conclusion remains that the generally invariant Lagrangian is dependent on tensorial quantities only.

The maximal number of functionally independent invariants is equal to the difference of the dimension of the manifold Q and of the rank of the discussed system of vector fields $\Sigma^{i,jkl}$, $\Sigma^{i,jk}$, Σ_{ij}^{+}, Σ_{ij}^{-} in their maximal points. In the case where the dimension of basic manifold $n = 1$, this rank is clearly maximal. In the case $n \geqslant 3$, maximality of the rank (at least in the neighborhood of some points) was proved for the typical fiber $T_n^2(R^{n*} \odot R^{n*})$ (Krupka, 1976). Our "extension" of the typical fiber cannot change this fact.

It remains to discuss the case $n = 2$. For the typical fiber $T_2^2(R^{2*} \odot R^{2*})$ it holds $\Xi_{ij}^{-} = 0$ and the rank of the system $\Xi^{i,jkl}$, $\Xi^{i,jk}$, Ξ_{ij}^{+}, Ξ_{ij}^{-} is smaller by 1 than the maximal one; as, however, Σ_{ij}^{-} is not identically zero, the rank of the system (3.2) for typical fiber Q will be maximal in this case as well.

It follows therefore that the maximal number of functionally independent invariants is

$$X = \tfrac{1}{12}n(n-1)(n-2)(n+3) + k[1 + n + \tfrac{1}{2}n(n+1)] \quad (4.2)$$

where k is the number of independent components of discussed (nonmetrical) fields. Let us note that in the case $k = 0$ (the metrical field only) the expression presented is not valid for $n = 2$.

Preceding results can be also applied to the case where $\mathscr{L}$ is dependent on the first derivatives of nonmetrical fields only. The identities (3.3) remain then valid after omitting the members with second derivatives of the nonmetrical fields. The number of functionally independent invariants will be smaller as (4.2) by $\frac{1}{2}nk(n+1)$.

However, in relativistic theories it is often supposed that $\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2$, where $\mathscr{L}_1 = \mathscr{L}_1(g_{ij}, g_{ij,k}, g_{ij,kl})$ and $\mathscr{L}_2 = \mathscr{L}_2(g_{ij}, g_{ij,k}, A_i, A_{i,j})$. Here, $\mathscr{L}_1$ is the "gravitational" and $\mathscr{L}_2$ the "material" part of the Lagrangian. The independency of $\mathscr{L}_2$ on the second derivatives of the metrical tensor field can be considered as the manifestation of the so-called principle of minimal gravitational coupling. In this case we arrive at the conclusion that $\mathscr{L}_2$ depends only on g_{ij}, A_i, $A_{i;j}$ in the case of the vector field (and analogically for an arbitrary number of tensor fields).

Let us reflect on the number of independent invariants of g_{ij}, A_i, $A_{i;j}$ (i.e., independent Lagrangians of the type $\mathscr{L}_2$). We choose on the manifold

$T_n^1(R^{n*} \odot R^{n*}) \times T_n^1(R^{n*})$ such points, where $A_{i;j} = \delta_{ij}$, $A_i = 0$, $g_{ij} = 0$, and $g_{ii} \neq g_{jj}$ for $i \neq j$. Then we obtain from (3.2), (3.1)

$$4\Sigma_{\overline{ij}} = (g_{ii} - g_{jj})\left(\frac{\partial}{\partial A_{i,j}} + \frac{\partial}{\partial A_{j,i}}\right)$$

As $i > j$, it is clear that the system of the vector fields $\Sigma_{\overline{ij}}$ has the maximal rank in the neighborhood of the points considered.

In the case of tensor field of the second degree let us choose on the manifold $T_n^1(R^{n*} \odot R^{n*}) \times T_n^1(R^{n*} \otimes R^{n*})$ such a point, where $A_{ij} = \delta_{ij}$, $A_{i;j} = 0$, $g_{ij} = 0$, and $g_{ii} \neq g_{jj}$ for $i \neq j$. It follows from (3.2), (4.1)

$$4\Sigma_{\overline{ij}} = (g_{ii} - g_{jj})\left(\frac{\partial}{\partial A_{ij}} + \frac{\partial}{\partial A_{ji}}\right)$$

so that the rank is again maximal. In the case of a tensor field of a higher degree let us choose $A_{i_1\cdots i_m} = \delta_{i_1 i_2}\cdot\delta_{i_1 i_3}\cdots\delta_{i_1 i_m}$, $A_{i\cdots i_m;k} = 0$, $g_{ij} = 0$ for $i \neq j$ and we obtain

$$4\Sigma_{\overline{ij}} = \left(\frac{\partial}{\partial T_{ij\ldots j}} + \frac{\partial}{\partial T_{jij\cdots j}} + \cdots + \frac{\partial}{\partial T_{j\ldots ji}}\right) g_{ii} - \left(\frac{\partial}{\partial T_{ji\cdots i}} + \frac{\partial}{\partial T_{iji\cdots i}} + \cdots + \frac{\partial}{\partial T_{i\ldots ij}}\right) g_{jj}$$

Clearly the rank of the system is again maximal.

The maximal number of functionally independent invariants of tensor fields and the metric of first degree (with the exception of scalar fields only) is consequently

$$k(n + 1) - \tfrac{1}{2}n(n - 1)$$

where k is the number of independent components of the fields studied. For example, in the case of a vector field, where $k = n$, this number is equal to $\frac{1}{2}n(n + 3)$, which in the physically significant case $n = 4$ gives the result 14. [Let us note that according to the preceding argument this number is equal to the number of invariants of vector (A_i), tensor of second degree ($A_{i;j}$), and the metrical tensor g_{ij}.]

For the case of scalar field φ it holds

$$\Sigma_{\overline{ij}} = (g_{ik}\varphi_{,j} - g_{jk}\varphi_{,i})\frac{\partial}{\partial \varphi_{,k}}$$

In this case the number of invariants is equal to the number of invariants of scalar φ, vector ($\varphi_{,i}$), and the metrical tensor g_{ij}, i.e., it is equal to 2. Finally, let us consider the case of the k scalar fields. Then the number of invariants is equal to the number of invariants of k vector fields $\varphi_{,i}^{(1)}, \ldots, \varphi_{,i}^{(k)}$

and the metrical tensor g_{ik} increased by the number of scalar fields considered, i.e., $\frac{1}{2}k(k + 3)$ for $k \leqslant n$ and $k(n + 1) - \frac{1}{2}n(n - 1)$ for $k \geqslant n$.

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